

- I. introduce PD stuff.
- ~~II~~ II. Define crystalline coh. "by hand".
- III. Define crystalline site.

Defn. A PD ring is a triple  $(A, I, \gamma)$  where  $I \subseteq A$  is an ideal.

and  $\gamma$  is a system of set maps  $\gamma_i: I \rightarrow A$  for  $i \geq 1$ , s.t.

0) (convention)  $\gamma_0(x) = 1$ ,  $\gamma_{-i}(x) = 0 \quad \forall i \geq 1$ .

1)  $\gamma_i(x) = x$ .  $\gamma_i(x) \in I$  if  $i \geq 1$ .

2)  $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x) \gamma_j(y)$ ,  $\forall x, y \in I$

3)  $\forall \lambda \in A, x \in I, \gamma_k(\lambda x) = \lambda^k \gamma_k(x)$ .

4)  $x \in I, \gamma_i(x) \gamma_j(x) = \binom{i+j}{i, j} \gamma_{i+j}(x)$  where  $\binom{i+j}{i, j} = \frac{(i+j)!}{i! j!}$ .

5)  $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p} \gamma_{pq}(x)$ .

Remark.

•  $\gamma$  is said a PD structure on  $I$ .

• Morally " $\gamma_i(x) = \frac{x^i}{i!}$ ". If  $\gamma$  is obvious from context, we denote  $\gamma_i(x) = x^i$ .

• 2) + 4)  $\Rightarrow n! \gamma_n(x) = x^n \quad \forall n \geq 0$ . & 3)  $\Rightarrow \gamma_k(0) = 0 \quad \forall k \geq 1$ .

morphism  $(A, I, \gamma) \rightarrow (B, J, \delta)$  is what you think it's.

e.g.  $\mathbb{O}$ -DVR, w/  $p = u \cdot \pi^e$  w/  $e \leq p-1$  for all prime  $p$ .

$(A, I, \gamma)$

Lemmata • if  $m \cdot A = 0$  for some  $m \in \mathbb{N} \Rightarrow I$  is locally nilpotent

• if  $J \subseteq I$  is a PD subideal. (i.e.  $\gamma_r(j) \in J \quad \forall j \in J, r \geq 1$ ).

then  $(A/J, I/J, \bar{\gamma})$  is also a PD structure.

&  $A \twoheadrightarrow A/J$  is a PD morphism.

• direct limit of PD w/ PD morphisms is still PD,  
↑  
transition

• If  $A \rightarrow B$ ,  $I \subseteq B$ ,  $J \subseteq C$  s.t.  $B \xrightarrow{\pi} B/I$   $C \xrightarrow{\pi} C/J$   
 $\downarrow$   $C$  then  $K = \ker(B \otimes_A C \rightarrow B/I \otimes_A C/J)$  has a  
 unique PD structure compatible w/  $(B, I)$  &  $(C, J)$ .

•  $I$  is a PD ideal  $\Rightarrow I^n \subseteq I$  PD subideal  
 e.g.  $(W(k), (p)) \Rightarrow (W_n(k), (\overline{p}))$  has a natural PD structure.

• Let  $A$  be a ring,  $M$  be an  $A$ -module.

$\exists \Gamma_A(M) = A \oplus M \oplus \Gamma_2(M) \oplus \dots$  w/  $\Gamma_A^+(M) = M \oplus \Gamma_2(M) \oplus \dots$   
 having PD structure  $\gamma$  s.t.

$$\text{Hom}_A(M, J) = \text{Hom}_{\text{PD}}(\Gamma_A(M), \Gamma_A^+(M), \gamma), (B, J, \delta)$$

$\forall$  PD ~~ring~~  $A$ -alg.  $(B, J, \delta)$ .

~~with behavior~~ preserved under base change, colimit.  
 turn ~~finite direct sum~~ into ~~tensor~~  $\otimes$ .

if  $M = \bigoplus_I A \cdot x_i$ .  $\Gamma_A(M) =: A \langle x_i ; i \in I \rangle$  is called  
 the divided polynomial alg. in  $\{x_i, i \in I\}$

Defn. ① <sup>given</sup>  $(A, I, \gamma)$ ,  $A \rightarrow B$ ,  
 we say  $\gamma$  extends to  $B$  if  $\exists \bar{\gamma}$  on  $I \cdot B$  s.t.

$(A, I, \gamma) \rightarrow (B, IB, \bar{\gamma})$  is a PD morphism.

②  $(A, I, \gamma)$ .  $A \rightarrow B$  is said compatible w/ PD structure if  
 Given  $(B, J, \delta)$ .  $\gamma$  extends to  $\bar{\gamma}$  &  $\bar{\gamma} = \delta$  on  $IB \cap J$ .

equivalently  $\exists$  PD structure  $\alpha$  on  $IB + J$  s.t.

$$(A, I, \gamma) \xrightarrow{\text{PD}} (B, IB + J, \alpha) \xleftarrow{\text{PD}} (B, J, \delta)$$

Thm. (PD envelope). <sup>Given</sup>  $(A, I, \gamma)$ ,  $A \rightarrow B$ ,  $J \subseteq B$  ideal.

Then  $\exists!$  a  $B$ -alg.  $D_{B, \gamma}(J)$  w/ PD ideal  $(\bar{J}, [n])$ .

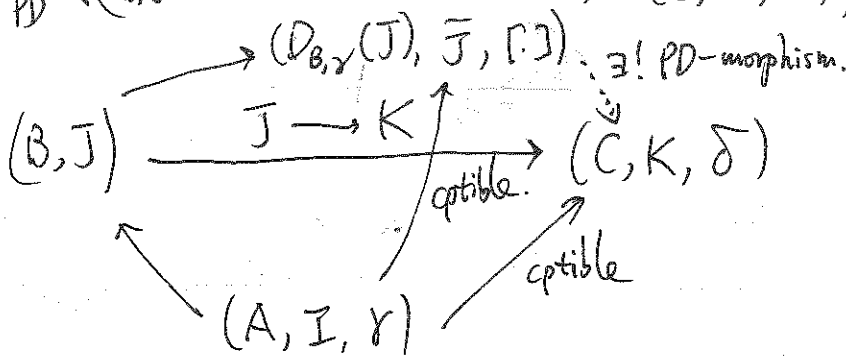
s.t. ①  $J \cdot D_{B, \gamma}(J) \subseteq \bar{J}$

②  $[\cdot]$  is cptible w/

$$(A, I, \gamma) \longrightarrow (D_{B, \gamma}(J), \bar{J}, [\cdot])$$

③ universal property:  $\forall (A, I, \gamma) \xrightarrow{\text{cptible}} (C, K, \delta)$ .

$$\text{Hom}_{\text{PD}}((D_{B, \gamma}(J), \bar{J}, [\cdot]), (C, K, \delta)) = \text{Hom}_{A_0}((B, J), (C, K)).$$



sketch of pf:  $D_{B, \gamma}(J) \cong \Gamma_B(IB+J) / \sim$  all the relations you can think of.

w/  $\bar{J}$  = PD generated by image of  $J$ . (in degree 1).

Rmk. We always have  $B/J \rightarrow D_{B, \gamma}(J) / \bar{J}$  i.e.  $\frac{D_{B, \gamma}(J)}{\bar{J}} \cong B/J$  iff  $\gamma$  extends to  $B/J$ .

Note that  $\gamma$  automatically extends if  $I$  is principal or  $IB \subseteq J$ .

II Let  $X_0$   
 $(A, I, \gamma)$   $\downarrow$  ~~is~~ lfp.  
 $S_0 = \text{Spec}(A/I) \longleftrightarrow \text{Spec}(A)$ .

Defn/Prop. (Assume  $X_0/S_0$  is separated, otherwise do a hypercover.)

0) Choose affine covering  $X_0 = \bigcup \text{Spec}(A_i)$ .

1)  $\forall i$ , choose smooth  $A$ -alg.  $B_i$  w/  $J_i \rightarrow B_i \twoheadrightarrow A_i$

2)  $\forall i_0, i_1, \dots, i_k$ , ~~form~~ we have  $J_{i_0 \dots i_k} \rightarrow B_{i_0} \otimes_A B_{i_1} \otimes_A \dots \otimes_A B_{i_k}$

~~is~~ and define  $D_{i_0 \dots i_k} = D_{B_{i_0 \dots i_k}, \gamma}(J_{i_0 \dots i_k})$ .  $\downarrow$   
 $A_{i_0} \cap A_{i_1} \cap \dots \cap A_{i_k}$ .

3) Form the ~~is~~ cplx ~~is~~  $\Omega_{D_0/A}$ .

~~is~~ Čech-dR double

4)  $H_{\text{crys}}^i(X_0/A, \mathcal{O}) := H^i(\text{Tot}(\Omega_{D_0/A}))$ .

which is well-defined!

Example. Let  $\mathbb{A}^1_{\mathbb{F}_k}$   $\downarrow$   $\text{Spec } k \longleftrightarrow \text{Spec } W_n(k)$  where  $k$  - perfect field of char.  $p > 0$ .

0)  $\mathbb{A}^1_{\mathbb{F}_k} = \cup A'_k$ .  $A_0 = k[x]$   $A_i = k[y]$   $A_{01} = k[x^{\neq 1}]$

1)  $B_0 = W_n[x]$   $B_i = W_n[y]$   $B_{01} = W_n[x, y]$ .

2)  $D_0 = W_n[x]$   $D_i = W_n[y]$ .

$D_{01} = W_n[x, y], (xy-1)^{[n]} / \sim (= W_n[x^{\neq 1}] \langle t \rangle)$ .

$W_n x^{p^n-1} dx$   
 $W_n y^{p^n-1} dy$

$W_n \oplus W_n \rightarrow W_n$

call  $xy-1 = t$ .  
 then  $t^{p^n} = 0$ , hence  $x$  invertible  
 &  $y = x^{-1} \cdot (1+t)$ .

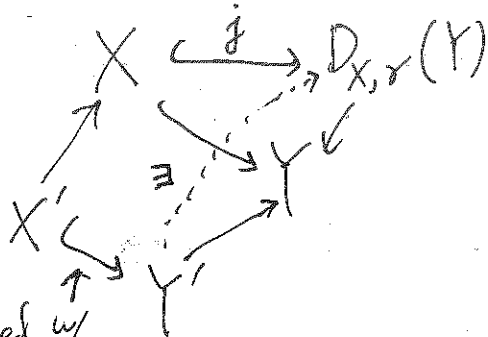
Computation in Brian's talk essentially shows...  
 $\frac{dx}{x}$ !

III. The discussion of PD stuff globalizes *mutatis mutandis*.

$$(S, \mathcal{I}, \gamma) \quad (\mathcal{X}, \text{quasi-coherent sheaf of ideals}) \quad v(\mathcal{J}) = \begin{array}{ccc} X & \xrightarrow{\text{closed}} & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

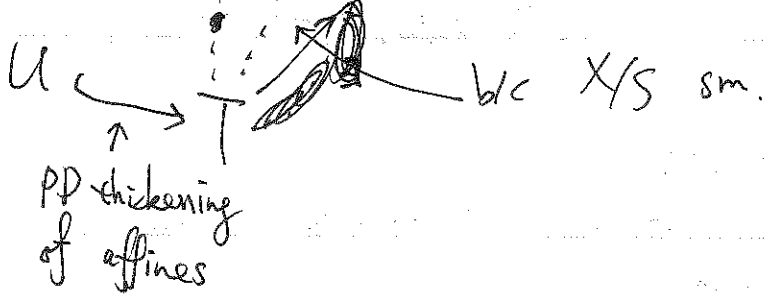
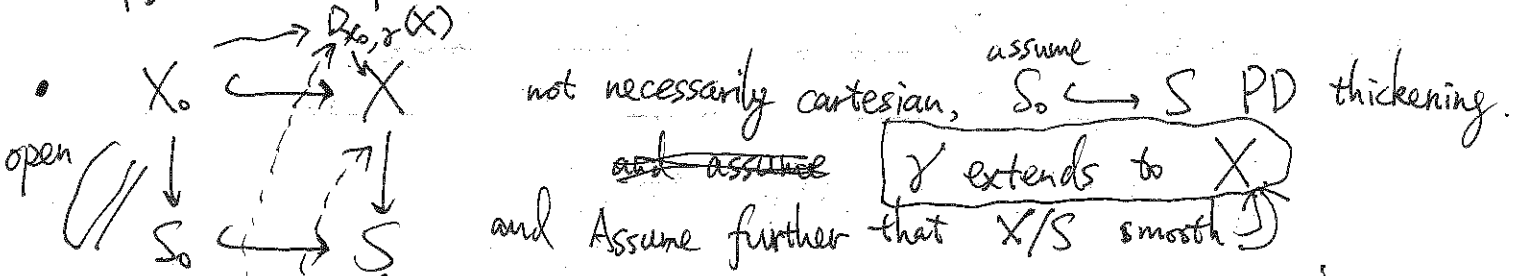
Defn.  $D_{X, \gamma}(Y) := \text{Spec}_Y D_{O_{X, \gamma}}(\mathcal{J})$ .

Rmk. So if  $\gamma$  extends to  $X$ , then



$(S, \mathcal{I}, \gamma)$ .

if closed w/ PD structure compatible w/  $\gamma$ .



assume  $p^n \cdot \mathcal{O}_S = 0$  (\*) e.g.  $W_n(k)$   $X_0/S$   $(S, \mathcal{I}, \gamma)$  PD-scheme.  $Y$  extends to  $X_0$

Defn:  $(X_0/S)$  crys obj:  $\begin{array}{ccc} U & \xrightarrow{\text{open}} & T \\ X_0 & \longrightarrow & S \end{array}$  where  $U = V(\mathcal{J})$   
 note: (\*)  $\Rightarrow \mathcal{J}$  is locally nilpotent  $(\forall j \in \mathcal{J}, j^{p^n} = 0)$ .  $\mathcal{J} \subseteq \mathcal{O}_T$   $(T, \mathcal{J}, \delta)$  PD compatible w/  $\gamma$ .

morphism: 
$$\begin{array}{ccc} T' & \longrightarrow & T \\ \downarrow & & \downarrow \\ U' & \xrightarrow{\text{open}} & U \end{array} \quad (T', \mathcal{J}', \delta') \longrightarrow (T, \mathcal{J}, \delta)$$
 PD-morphism.

covering:  $\{T_i \rightarrow T\}$  is an open immersion & jointly cover  $|T|_{\text{Zar}}$ .

Caution: It doesn't have ~~enough~~ final obj.

Prop/Defn. A sheaf  $\mathcal{F} \in \text{Sh}((X_0/S)_{\text{crys}}) \iff$   
 a law associating each  $(U, T, \delta) \text{ w/ } \mathcal{F}_T \in \text{Sh}(T_{\text{Zar}})$  s.t.

$$\forall (U', T', \delta') \xrightarrow{u} (U, T, \delta), \exists \rho_u: u^* \mathcal{F}_T \rightarrow \mathcal{F}_{T'}$$

satisfying composition law

and if  $T' \xrightarrow[\text{open}]{\alpha} T$ , then  $\rho_u$  is an isom.

e.g. (1)  $(U, T, \delta) \longmapsto \mathcal{O}_T$ , denoted  $\mathcal{O}_{X_0/S}$ .

(2)  $(U, T, \delta) \longmapsto \mathcal{O}_U$ , denoted  $i_{X*} \mathcal{O}_{X_0}$ .

(3)  $0 \rightarrow \mathcal{J}_{X/S} \rightarrow \mathcal{O}_{X_0/S} \rightarrow i_{X*} \mathcal{O}_{X_0} \rightarrow 0$  exact.

$$\mathcal{J}_{X/S}(U, T, \delta) = \ker(\mathcal{O}_T \rightarrow \mathcal{O}_U).$$

Remk. The prop/Defn above implies that  $\text{Sh}((X_0/S)_{\text{crys}})$  has enough points.

Prop. (1) representable objects are sheaves.

(1) absolute finite product / inverse limit over finite nonempty index set of representable sheaves are representable.

(2) 
$$\begin{array}{ccc} X_0' & \xrightarrow{g} & X_0 \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\text{PD}} & S \end{array} \rightsquigarrow \text{get } g_{\text{crys},*} \text{ \& } g_{\text{crys}}^{-1}$$

$$\text{Sh}(X_0'/S') \longrightarrow \text{Sh}(X_0/S) \text{ morphism of topoi.}$$

$g_{\text{crys}}^{-1}(\mathcal{F})(U, T, \delta) = \text{Hom}_{g\text{-PD}}(g_{\text{crys}}^{-1}(U, T, \delta), \mathcal{F})$

$g_{\text{crys},*}(\mathcal{F})(U, T, \delta) = \text{Hom}(g_{\text{crys}}^{-1}(U, T, \delta), \mathcal{F})$   
 $= \Gamma((X_0'/S')_{\text{crys}}|_{(U, T, \delta)}, \mathcal{F}|_{(U, T, \delta)})$

•  $g_{\text{crys}}^{-1}(F)(U', T', \delta') = \text{colim}_{T' \rightarrow g_{\text{crys}}^{-1}(T)} F(T)$

(3)  $\text{Sh}^{\#}(X_0/\mathbb{Z}_{\text{ar}}) \xrightarrow{i} \text{Sh}(X_0/S)_{\text{crys}} \xrightarrow{u} \text{Sh}(X_0/\mathbb{Z}_{\text{ar}})$   
 $\text{id.}$

$i$  is induced by  $(U, T, \delta) \longleftrightarrow U$ .

In particular  $i_* \mathcal{O}_X(U, T, \delta) = \text{Hom}_{\text{crys}}(U, T, \delta)$ ,  $i_* \mathcal{O}_X = \text{Hom}_{\mathbb{Z}_{\text{ar}}}(U, \mathcal{O}_X) = \mathcal{O}_X(U)$ . Hence agrees w/ previous notation.

- $u_*(F)(U) = \Gamma(U/S)_{\text{crys}}, F$ .
- $u^{-1}(F)(U, T, \delta) = E(U)$ .

Thm.  $X_0 \xrightarrow{i} X$  Then  $H^i(X/S)_{\text{crys}}, \mathcal{O}_{X/S} \cong H^i(X_0/S)_{\text{crys}}, \mathcal{O}_{X_0/S}$ .  
 $\downarrow \quad \# \quad \downarrow$   
 $S_0 \xrightarrow[\text{VCI}]{\text{PD}} S$  If: ①  $i_{\text{crys}}^{-1} T$  is represented by  $(U \cap X_0 = U_0, T, \bar{\delta})$   
 We claim  $(U, T, \delta)$  in  $(X_0/S)_{\text{crys}}$ .

②  $i_{\text{crys},*}$  is exact

③  $i_{\text{crys},*}(\mathcal{O}_{X_0/S}) \cong \mathcal{O}_{X/S}$ .

④ Leray spectral sequence for  $(\Gamma \circ i_{\text{crys}})$  implies what we want.

④ we just have to verify  $R\Gamma_{X/S}(i_{\text{crys},*} F) = R\Gamma_{X_0/S}(F)$

①  $\Rightarrow$  ③:  $i_{\text{crys},*}(\mathcal{O}_{X_0/S})(T)$   
 $= \text{Hom}(i^{-1} h_T, \mathcal{O}_{X_0/S})$   
 $= \mathcal{O}_{X_0/S}(U_0, T, \bar{\delta}) = \mathcal{O}_T(T) = \mathcal{O}_{X/S}(T)$ .

$R\text{Hom}(\mathcal{O}_{X/S}, i_{\text{crys},*} F) \quad R\text{Hom}(\mathcal{O}_{X_0/S}, F)$

$R\text{Hom}(i_{\text{crys}}^{-1} \mathcal{O}_{X/S}, F)$  since  $i_{\text{crys}}^{-1}(\mathcal{O}_{X/S}) = \mathcal{O}_{X_0/S}$

as it preserves finite limit &

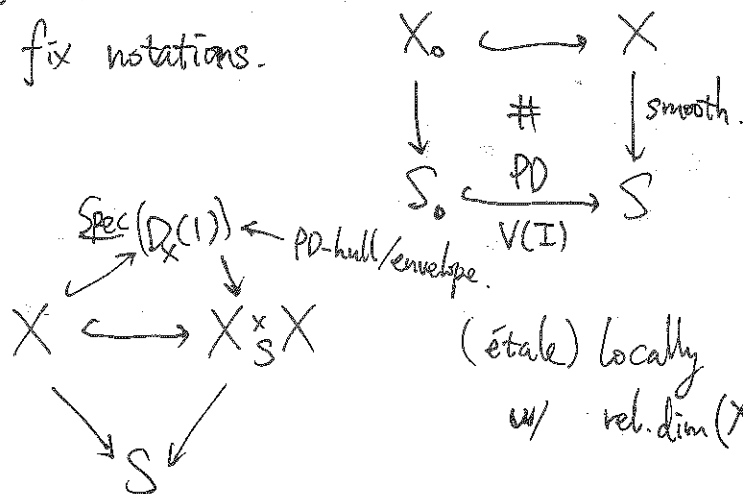
final obj. is empty limit.

①  $\Rightarrow$  ②: follows similarly. ( $i^{-1}$  doesn't change the "T"-part, and Crys. sheaf has a Zariski interpretation)

①: by definition

Now we're ready to prove the famous comparison thm to de Rham coh. assuming appearance of a smooth lift.

Let us fix notations.



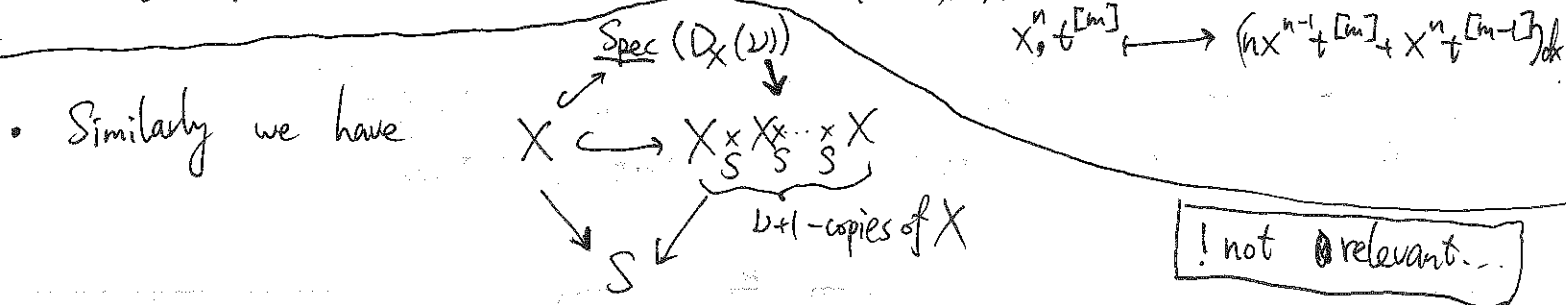
(étale) locally  $D_X(1)$  is divided power dg. over  $X$  w/  $\text{rel. dim}(X/S)$  many variables.

Fact (To be explained in the next 2 talks):  $\mathcal{O}_{X_0/S} \xrightarrow{\text{qis}} \mathcal{L}(\Omega_{X/S}^\bullet) := D_X(1) \otimes_{\mathcal{O}_X} \Omega_{X/S}^\bullet$

Cautions: The connecting morphism on the RHS is subtle.

e.g.  $S = (A, I, \gamma)$ .  $X = \text{Spec } A[x]$ . Then  $D_X(1) = A[x]\langle t \rangle$ .

We have exact sequence  $0 \rightarrow A[x] \rightarrow A[x]\langle t \rangle \rightarrow A[x]\langle t \rangle dx \rightarrow 0$   
 (this only verifies on  $(X, X, \mathcal{O}) \dots$ )



The sheaf  $h_X =: \tilde{X}$  has the property (due to smoothness of  $X/S$ ) that

Also  $\tilde{X} \times_e \tilde{X} \times_e \dots \times_e \tilde{X} = \text{Spec}(D_X(1)) \rightarrow e$  is a covering/epimorphism.

Hence for any (bounded) cplx in  $\text{Sh}(X/S)_{\text{qis}}$ , we have ~~may use the double cplx~~  $K^*$

$$K^* \xrightarrow{\text{qis}} (Rj_{X,*} j_X^{-1} K^* \rightarrow Rj_{X,*} j_X^{-1} K^* \rightarrow \dots) \quad (\text{notation to be explained}) \\
 =: \check{C}A_{\tilde{X}}(K^*)$$





~~where in the last step, we have used:~~

~~$$\text{fixing } k: \Omega_{X/S}^k \xrightarrow{q_{is}} \left( D(1) \otimes_X \Omega_{X/S}^k \rightarrow D(2) \otimes_X \Omega_{X/S}^k \rightarrow \dots \right)$$~~

~~(c.f. [Bhatt-dJ] Lemma 2.17)~~

Apply  $R\Gamma(X_{\text{Zar}}, -)$  to both sides, we get the following thm.

Thm 
$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow \# & \downarrow \text{sm.} & \\ S_0 & \hookrightarrow & S \end{array} \quad \text{we have } H^i((X_0/S)_{\text{crys}}, \mathcal{O}_{X_0/S}) \cong H_{\text{dR}}^i(X/S).$$

Back to Faltings's Defn:

$$\left\{ \coprod_i (\text{Spec } A_i, \text{Spec } B_i, \tilde{\gamma}) = T_i \right\} \longrightarrow e \text{ forms a covering/epimorphism in } \text{Sh}((X_0/S)_{\text{crys}})$$

$$\text{w/ } \prod_{i \in \{i_0, \dots, i_k\}} \tilde{T}_i = \left( \prod_I \text{Spec } A_i, \text{Spec } D_{B_{i_0 \dots i_k}, \tilde{\gamma}}(T_{i_0 \dots i_k}), \tilde{\gamma} \right) \sim$$

Hence 
$$\left( R\Gamma(\prod_I T_i / S)_{\text{crys}} \rightarrow R\Gamma(\prod_{ij} T_{ij}, \mathcal{O}_{X_0/S}) \rightarrow \dots \right)$$
  
 computes  $R\Gamma((X_0/S)_{\text{crys}}, \mathcal{O}_{X_0/S})_{S_{\text{crys}}}$ .

Then we use [Bhatt-dJ] Thm 2.12.

$$R\Gamma\left(\prod_I T_i / S\right)_{\text{crys}}, \mathcal{O}) \simeq \left( \mathcal{O} \rightarrow \Omega_D^1 \rightarrow \Omega_D^2 \rightarrow \dots \right)$$

where notations have to be understood appropriately...